A BINOMIAL DETERMINATE: A VERY PLAY IN THREE ACTS

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Scenery:

Matrix size and indexing: $i, j = 0..n$; n

The Tensor-Einstein summation conventions:

Superscript/Subscript for Matrix elements: For X^i_j i indexes the rows, j indexes the columns

i.e. using Einstein summation convention

$X^i_r Y^r_j = Z^i_j$ is matrix multiplication $\left[X^i_r\right]\left[Y^r_j\right] = \sum_{r=0..n} X^i_r Y^r_j = \left[Z^i_j\right]$

Characters:

Binomial coefficient: $\binom{i}{j} = \begin{cases} 0 & j > i \\ 0 & j < 0 \\ \dfrac{i!}{j!(i-j)!} & else \end{cases}$ [4]

$B : B^i_{j;k} = \binom{i+j+k}{i} = \binom{i+j+k}{j+k}$;

A: the minor corresponding to the cofactor B^0_0

P: the Pascal matrix$P^i_j = \binom{i}{j}$; $\widetilde{P}$ is the transpose $\widetilde{P}^i_j = \binom{j}{i}$[2, 1]

$P, \widetilde{P}$ are triangular, the diagonals elements are 1; thus the determinants are 1

Referring to [1] it is easy to show that. $P^{-1} = \binom{i}{j}(-1)^{i+j}$; $\widetilde{P}^{-1} = \binom{j}{i}(-1)^{i+j}$

S_k: the shifted Pascal Matrix $[S_k]^i_j = \binom{i+k}{j+k} = \binom{i+k}{i-j}$[1]

Refer ed to as D^{i+k}_{j+k} below.

S_k is triangular, the diagonal elements are 1; thus the determinate is 1

Referring to [1] it is easy to show that. $\left([S_k]^i_j\right)^{-1} = \binom{i+k}{j+k}(-1)^{i+j}$

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Vandermonde's Identity: $\binom{m+n}{r} = \sum_{l=0}^{l=r} \binom{m}{l}\binom{n}{r-l} = \sum_{l=0}^{l=r} \binom{m}{l}\binom{n}{l+n-r}$ [3]

The upper limit $l = r$ can be extended but yields 0 terms; as does $m, n < r, r-l$.

Summation Identity: $\binom{n+1}{r+1} = \sum_{m=r}^{n} \binom{m}{r} = \sum_{m=0}^{n-r} \binom{r+m}{r}$ [5]

Or for our problem $\binom{k+n+1}{k+1} = \sum_{r=0}^{n} \binom{k+r}{k}$

D: a matrix factor of B, an example of S_k, after being exposed by examining the evidence.

E : a matrix factor of B, an example of $\widetilde{P}$, after being exposed by examining the evidence

Plot: Find $det(A)$

Where $A^i_j = \left[\binom{i+j+k+2}{i+1}\right] = \left[\binom{i+j+k+2}{j+k+1}\right]$

Play:

Act 1: The cleavage of the Victim.

Using Vandermonde's identity to extract the top j we obtain

$$B^i_{j;k} = \binom{i+j+k}{i} = \sum_{l=0}^{l=r} \binom{j}{l}\binom{i+k}{l+k} = D^{i+k}_{l+k} \cdot \widetilde{P}^l_j = D \cdot \widetilde{P}$$

Act 2: Analysis of the body parts.

D^{i+k}_{r+k} is a shifted Pascal matrix

$\widetilde{P}^r_j$ is a transposed Pascal matrix

Since the determinants of $D, \widetilde{P}$ are one: $det(B) = 1$

$B^{-1} = \widetilde{P}^{-1} D^{-1}$

By standard matrix theory $(B^{-1})^0_0$ is the determinate of the minor corresponding to $A = B^0_0$

Act 3: The solution.

The $(B^{-1})^0_0$ term is given by $(\widetilde{P}^{-1})^0_r (D^{-1})^r_0$

$(\widetilde{P}^{-1})^0_r$ is a row vector of $(-1)^r$; so we are summing the column vector.

$\binom{r+k}{0+k}(-1)^r$

Thus $det(A) = \sum_{r=0}^{n} \binom{r}{0} \cdot \binom{r+k}{0+k} (-1)^{2r}$

Using the summation identity

$$det(A) = \binom{n+k+1}{k+1}$$

□

References

[1] L. Aceto and D. Trigiante. The matrices of Pascal and other greats. *The American Mathematical Monthly*, 108(3):232–245, 2001.

[2] Wikipedia. Pascal matrix — wikipedia, the free encyclopedia, 2009. [Online; accessed 10-January-2010].

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[4] Wikipedia. Binomial coefficient — wikipedia, the free encyclopedia, 2010. [Online; accessed 10-January-2010].

[5] Wikipedia. Binomial coefficient-equation (10) — wikipedia, the free encyclopedia, 2010. [Online; accessed 10-January-2010].